SADLER MATHEMATICS METHODS UNIT 2

WORKED SOLUTIONS

Chapter 2 Exponential functions

Exercise 2A

Question 1

x	0	1	2	3	4	5
y	1	3	9	27	81	243

Question 2

x	0	1	2	3	4	5
y	1	7	49	343	2401	16 807

Question 3

x	0	1	2	3	4	5
y	1.5	3	6	12	24	48

Question 4

x	0	1	2	3	4	5
y	1.75	14	112	896	7168	57 344

Question 5

x	0	1	2	3	4	5
y	2	4	8	16	32	64

Question 6

x	0	1	2	3	4	5
y	10	40	160	640	2560	10 240

Question 7

x	0	1	2	3	4
y	1	2	5	10	17

1	3	5	7

2	2	2

A constant second difference of 2 indicates a quadratic relationship with $a = 1 \Rightarrow y = x^2 + bx + c$

When $x = 0, y = 1 \Rightarrow c = 1$

When $x = 1$

$$\begin{aligned} y = 1(1)^2 + b(1) + 1 &= 2 \\ 2 + b &= 2 \\ b &= 0 \end{aligned}$$

Required equation $y = x^2 + 1$

Question 8

x	0	1	2	3	4
y	1	4	16	64	256

× 4	× 4	× 4	× 4

A constant ratio of 4 indicates the relationship is exponential, base 4. $\Rightarrow y = a \times 4^x$

When $x = 0$,
$y = a.(4^0) = 1$
$a = 1$

Required equation $y = 4^x$

Question 9

x	0	1	2	3	4
y	3	5	7	9	11

2	2	2	2

A constant first difference of 2 indicates a linear relationship with a gradient of 2. $\Rightarrow y = 2x + c$

When $x = 0, y = 3 \Rightarrow c = 3$

Required equation $y = 2x + 3$

Question 10

x	0	1	2	3	4
y	0	2	8	18	32

2	6	10	14

4	4	4

A constant second difference of 4 indicates a quadratic relationship with $a = 2. \Rightarrow y = 2x^2 + bx + c$

When $x = 0, y = 0 \Rightarrow c = 0$

When $x = 1$,

$$y = 2(1)^2 + b(1) = 2$$
$$2 + b = 2$$
$$b = 0$$

Required equation $y = 2x^2$

Question 11

x	0	1	2	3	4
y	1.5	12	96	768	6144

× 8	× 8	× 8	× 8

A constant ratio of 8 indicates the relationship is exponential, base 8. $\Rightarrow y = a \times 8^x$

When $x = 0$,

$$y = a.(8^0) = 1.5$$
$$a = 1.5$$

Required equation $y = 1.5 \times 8^x$

Question 12

x	0	1	2	3	4
y	1	5	25	125	625

× 5	× 5	× 5	× 5

A constant ratio of 5 indicates the relationship is exponential, base 5. $\Rightarrow y = a \times 5^x$

When $x = 0$,

$y = a.(5^0) = 1$

$a = 1$

Required equation $y = 5^x$

Question 13

x	0	1	2	3	4
y	0	2	6	12	20

2	4	6	8

2	2	2

A constant second difference of 2 indicates a quadratic relationship with $a = 1$. $\Rightarrow y = x^2 + bx + c$

When $x = 0, y = 0 \Rightarrow c = 0$

When $x = 1$,

$$\begin{aligned} y = 1(1)^2 + b(1) &= 2 \\ 1 + b &= 2 \\ b &= 1 \end{aligned}$$

Required equation $y = x^2 + x$

Question 14

x	0	1	2	3	4
y	1	6	36	216	1296

× 6	× 6	× 6	× 6

A constant ratio of 6 indicates the relationship is exponential, base 6. $\Rightarrow y = a \times 6^x$

When $x = 0$,
$y = a.(6^0) = 1$
$a = 1$

Required equation $y = 6^x$

Question 15

x	0	1	2	3	4
y	3	6	12	24	48

× 2	× 2	× 2	× 2

A constant ratio of 2 indicates the relationship is exponential, base 2. $\Rightarrow y = a \times 2^x$

When $x = 0$,
$y = a.(2^0) = 3$
$a = 3$

Required equation $y = 3 \times 2^x$

Question 16

x	1	2	3	4	5
y	60	30	20	15	12

No constant differences or ratios present. A reciprocal relationship exists $\Rightarrow xy = 60$.

Question 17

x	0		1		2		3		4
y	1		2		9		28		65
		1		7		19		37	
			6		12		18		
				6		6			

A constant third difference of 6 indicates a cubic relationship with $a = 1. \Rightarrow y = ax^3 + bx^2 + cx + d$.
(See Miscellaneous Exercise 1 Question 1)

When $x = 0, y = 1 \Rightarrow d = 1$

We currently have $y = x^3 + bx^2 + cx + 1$

When $x = 1$,

$y = 1 + b + c + 1 = 2$

$b + c = 0$

When $x = 2$,

$y = 8 + 4b + 2c + 1 = 9$

$4b + 2c = 0$

By simultaneous equations or CP,

$b = 0, c = 0$

Required equation $y = x^3 + 1$

Alternatively students may have noticed the numbers are all one more than the cubic numbers 1, 8, 27, 64…

Question 18

x	0	1	2	3	4
y	20	17	14	11	8

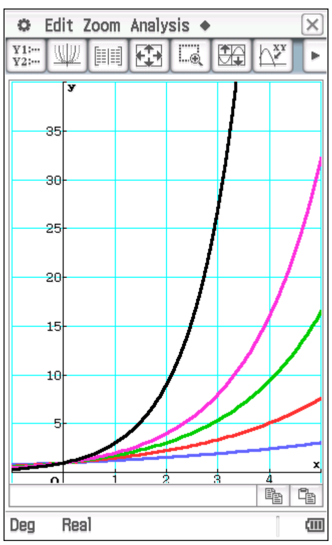
–3	–3	–3	–3

A constant first difference decreasing by 3 indicates a linear relationship with a gradient of -3.
$\Rightarrow y = -3x + c$

When $x = 0, y = 20 \Rightarrow c = 20$

Required equation $y = -3x + 20$

Question 19



All graphs pass through (0, 1).

The graphs share the same basic shape.

As the *x* values increase, so do the *y* values without limit.

As the *x* values decrease so do the *y* values,

approaching but never touching the *x*-axis.

We say it is asymptotic to the *x*-axis.

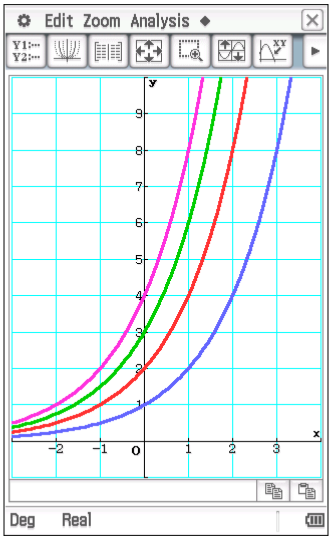
The values are always positive.

As the value of *a* increases,

so does the rate at which the *y* values increase.

The graph becomes steeper as *a* increases.

Question 20

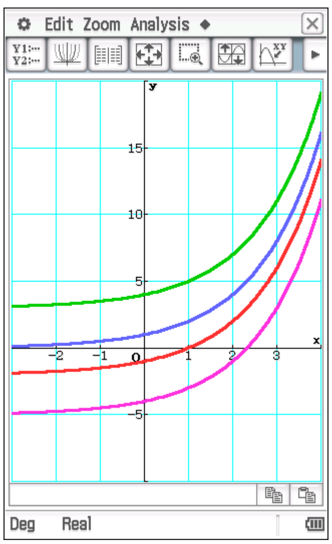


Changing the value of a changes the y-intercept

of the function to (0, a).

It appears to stretch or vertically dilate the original function.

Question 21



The value of k translates the graph vertically.

Graphs of the form $y = a^x + k$ translate $y = a^x$ k units upwards.

Graphs of the form $y = a^x - k$ translate $y = a^x$ k units downwards.

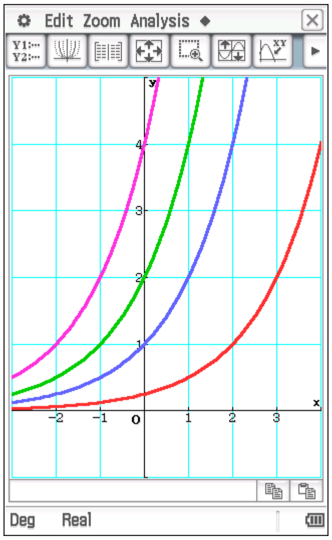
The distance of the y-intercept from (0, 1) indicates the value of k.

Similarly the distance of the horizontal asymptote from x-axis

will indicate the value of k.

It is possible for these graphs to have an x-intercept.

Question 22



Changing the value of k translates the graph horizontally.

Graphs of the form $y = a^{x+k}$ translate $y = a^x$ k units left.

Graphs of the form $y = a^{x-k}$ translate $y = a^x$ k units right.

Question 23

a

x	0	1	2	3
y	1	2	4	8

The y values are doubling $\Rightarrow y = 2^x$

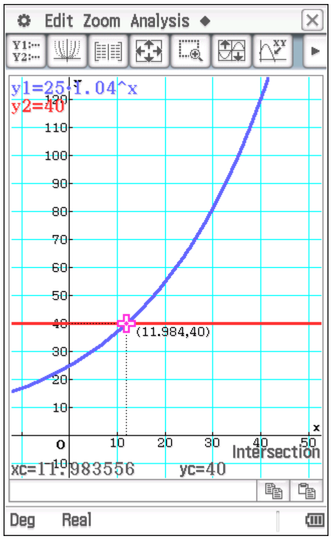
b

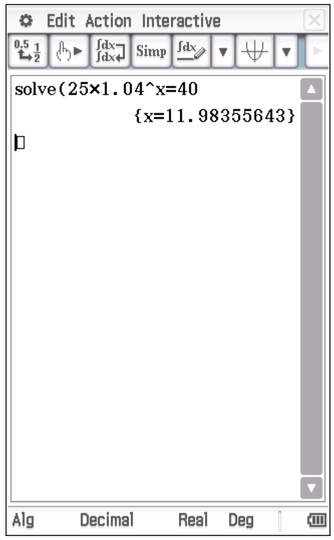
x	0	1	2
y	1	3	9

The y values are tripling $\Rightarrow y = 3^x$

Question 24

a

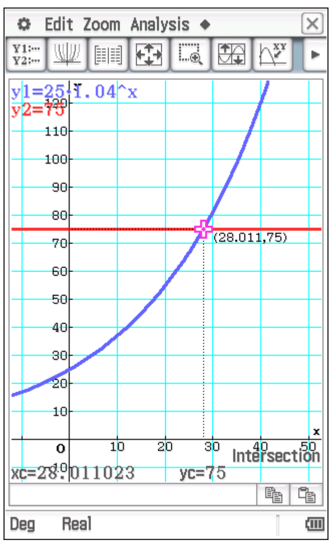


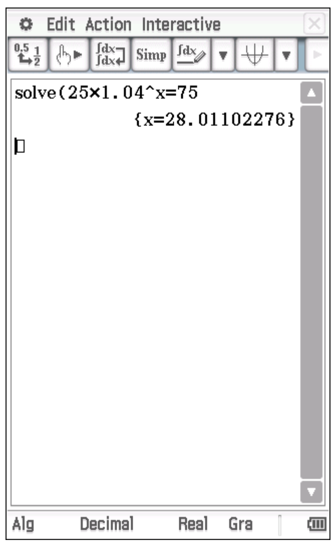


$x = 11.984$

In approximately 12 years.

b

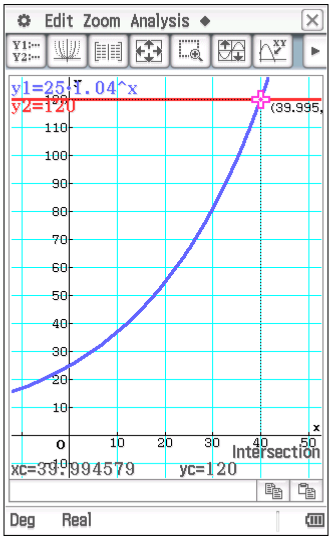


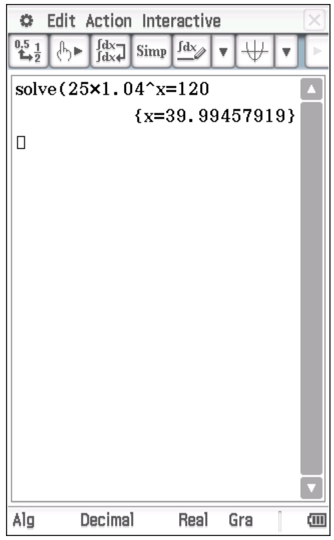


$x = 28.011$

In approximately 28 years.

c





$x = 39.995$

In approximately 40 years.

Question 25

a

x	2	3
y	1	3

y values are tripling indicating $y = 3^x$.

The point (2, 1) indicates a horizontal translation of 2 units to the right $\Rightarrow y = 3^{x-2}$.

b

x	0	1	2	3
y	3	4	6	10

1	2	4

The differences between *y* values are 1, 2, 4 indicating $y = 2^x$.

The point (0, 3) indicates a vertical translation of 2 units $\Rightarrow y = 2^x + 2$.

c

x	2	3	4	5
y	1	2	4	8
		1	2	4

The y values are 1, 2, 4 indicating $y = 2^x$.

The point (2, 1) indicates a horizontal translation of 2 units to the right $\Rightarrow y = 2^{x-2}$.

d

x	0	1	2
y	–1	1	7
		2	6

The graph appears to have an asymptote at $y = -2$, suggesting $y = a^x - 2$.

If we translate the graph up two units, the y values become 1, 3, 9 clearly indicating $a = 3$.

$y = 3^x - 2$

e

x	–1	0	1
y	3	5	11

The graph appears to have an asymptote at $y = 2$, suggesting $y = a^x + 2$.

If we translate the graph down two units, the y values become 1, 3, 9 clearly indicating $a = 3$.

$y = 3^x + 2$

The point (–1, 3) suggests a horizontal translation of 1 unit left $\Rightarrow y = 3^{x+1} + 2$.

f

x	2	3	4	5
y	–1	0	2	6

The graph appears to have an asymptote at $y = -2$, suggesting $y = a^x - 2$.

If we translate the graph up two units, the y values become 1, 2, 4, 8 clearly indicating a = 2.

$y = 2^x - 2$

The point (2, –1) suggests a horizontal translation of 2 units right $\Rightarrow y = 2^{x-2} - 2$.

Question 1

Consider the period 1995 to 2000 and working with the population in millions:

$$26 = 18r^5$$
$$r^5 = \frac{26}{18}$$
$$r = \sqrt[5]{\frac{26}{18}}$$
$$= 1.076$$
$$\approx 1.08$$

Consider the period 2000 to 2010:

$$56 = 26r^{10}$$
$$r^{10} = \frac{56}{26}$$
$$r = \sqrt[10]{\frac{56}{26}}$$
$$= 1.079$$
$$\approx 1.08$$

Question 2

Consider the period 2000 to 2007:

$$\begin{aligned}9000 &= 12400r^7\\ r^7 &= \frac{9000}{12400}\\ r &= \sqrt[7]{\frac{9000}{12400}}\\ &= 0.955\end{aligned}$$

$$\begin{aligned}7500 &= 9000r^7\\ r^4 &= \frac{7500}{9000}\\ r &= \sqrt[4]{\frac{7500}{9000}}\\ &= 0.955\end{aligned}$$

A ratio of 0.955 is equivalent to a 4.5% decrease each year.

Question 3

$$\frac{45.8}{45} = 1.0178$$

$$\frac{46.6}{45.8} = 1.0175$$

$$\frac{47.5}{46.6} = 1.0193$$

Average of three growth rates ≈ 1.0182, so we will assume an 18% growth rate overall.

In 2027, 14 years of growth has occurred.

$$\begin{aligned}P &= 47.5 \times 1.018^{14}\\ &= 60.98 \text{ million}\end{aligned}$$

Approximately 61 million.

Question 4

$$\frac{16500}{18100} = 0.917$$
$$\frac{15200}{16500} = 0.921$$
$$\frac{14000}{15200} = 0.921$$

Let us assume a growth rate of 92% on each year, (decreasing by 8% each year).

IF we consider 2010 to be $t = 0$, 2023 can be represented as $t = 13$.

$$\begin{aligned} P &= 18 \times 0.92^{13} \\ &= 6.1 \end{aligned}$$

We could expect approximately 6 100 animals.

Question 5

a $N = Ak^{N-1989}$

When $N = 1989$,

$80 = Ak^0$

$A = 80$

When $N = 1999$

$170 = 80k^{10}$

$$k^{10} = \frac{170}{80}$$

$$\begin{aligned} k &= \sqrt[10]{\frac{170}{80}} \\ &= \left(\frac{170}{80}\right)^{0.1} \\ &= 1.08 \end{aligned}$$

b 8% increase

c
$$\begin{aligned} N &= 80 \times 1.08^{(2024-1989)} \\ &= 80 \times 1.08^{35} \\ &\approx 1200 \end{aligned}$$

Question 6

$P = ak^n$

After 6 days $\Rightarrow 450 = ak^6$

After 4 days $\Rightarrow 530 = ak^5$

$$\frac{ak^6}{ak^5} = \frac{450}{530}$$
$$k = 0.85$$

$$530 = a(0.85)^5$$
$$a = \frac{530}{0.85^5}$$
$$= 1194$$

The initial population was approximately 1200 frogs.

Question 7

a 68

b 29

c

$$68 = ka^3$$
$$29 = ka^8$$
$$\frac{29}{68} = a^5$$
$$a = \sqrt[5]{\frac{29}{68}}$$
$$= 0.84$$

$$68 = k(0.84)^3$$
$$k \approx 115$$

d

$$P = 115 \times 0.84^0$$
$$= 115$$

e

$10 = 115(0.84)^t$

By classpad

$t = 14$

Question 8

a $P = ka^t$

The initial value is 80 $\Rightarrow k = 80$

When $t = 3, P = 62$

$62 = 80a^3$

$a^3 = \frac{62}{80}$

$a = \sqrt[3]{\frac{62}{80}}$

$= 0.92$

b

In 2017, $t = 3$

$P = 80(0.92)^3$

$= 27$

c $20 = 80(0.92)^t$

By classpad, $t = 16.6$

During the 17th year, 2021

Question 9

a When t=0, $P_A = 10\ 000$ and $P_B = 1000$

b $P_A = 10000(0.75)^3 = 4218.75$

$P_B = 1000(1.09)^3 = 1295.03$

$P_A = 4200$ and $P_B = 1300$

c $1000(0.75)^t = 1000(1.09)^t$

By classpad, $t = 6.2$

Question 10

a Initial value 850 $\Rightarrow k = 850$

When $t = 3, N = 630$

$$630 = 850a^3$$

$$a^3 = \frac{630}{850}$$

$$a = \sqrt[3]{\frac{630}{850}}$$

$$= 0.905$$

b $212.5 = 850(0.905)^t$

By classpad, $t = 13.88$

after 14 weeks

Miscellaneous exercise two

Question 1

a II

b IV

c III

d I

e III

f IV

g III

h I

Question 2

a
$$\begin{aligned} x^2 &= 49 \\ x &= \pm 7 \end{aligned}$$

b
$$\begin{aligned} x^2 &= 100 \\ x &= \pm 10 \end{aligned}$$

c
$$\begin{aligned} x^3 &= 1000 \\ x &= 10 \end{aligned}$$

d
$$\begin{aligned} 2^x &= 4 \\ &= 2^2 \\ x &= 2 \end{aligned}$$

e
$$\begin{aligned} 3^x &= 81 \\ &= 3^4 \\ x &= 4 \end{aligned}$$

f
$$\begin{aligned} 5^x + 11 &= 12 \\ 5^x &= 1 \\ &= 5^0 \\ x &= 0 \end{aligned}$$

g $$\begin{aligned} 6^x + 9 &= 225 \\ 6^x &= 216 \\ &= 6^3 \\ x &= 3 \end{aligned}$$

h $$\begin{aligned} 4^x &= \frac{1}{4} \\ &= 4^{-1} \\ x &= -1 \end{aligned}$$

i $$\begin{aligned} 4^x &= \frac{1}{16} \\ &= 4^{-2} \\ x &= -2 \end{aligned}$$

j $$\begin{aligned} 4^x &= \frac{1}{64} \\ &= 4^{-3} \\ x &= -3 \end{aligned}$$

k $$\begin{aligned} 2^x &= \frac{1}{2} \\ &= 2^{-1} \\ x &= -1 \end{aligned}$$

l $$\begin{aligned} 2^x &= \frac{1}{4} \\ &= 2^{-2} \\ x &= -2 \end{aligned}$$

m $$\begin{aligned} 2^x &= \frac{1}{8} \\ &= 2^{-3} \\ x &= -3 \end{aligned}$$

n $$\begin{aligned} &16x^4 = 400x^2 \\ &16x^4 - 400x^2 = 0 \\ &16x^2(x^2 - 25) = 0 \\ &x^2 = 0 \text{ or } x^2 - 25 = 0 \\ &x = 0 \qquad\qquad x = \pm 5 \end{aligned}$$

o

$$
\begin{aligned}
8^{2x+1} &= 4^{1-x} \\
(2^3)^{2x+1} &= (2^2)^{1-x} \\
2^{6x+3} &= 2^{2-2x} \\
6x+3 &= 2-2x \\
8x &= -1 \\
x &= -\frac{1}{8}
\end{aligned}
$$

p

$$
\begin{aligned}
\sqrt{50}x-\sqrt{18}x &= \sqrt{2} \\
5\sqrt{2}x-3\sqrt{2}x &= \sqrt{2} \\
2\sqrt{2}x &= \sqrt{2} \\
x &= \frac{\sqrt{2}}{2\sqrt{2}} \\
&= \frac{1}{2}
\end{aligned}
$$

q

$$
\begin{aligned}
5\sqrt{2}\sqrt{x}-3\sqrt{2}\sqrt{x} &= \sqrt{2} \\
2\sqrt{2}\sqrt{x} &= \sqrt{2} \\
\sqrt{x} &= \frac{\sqrt{2}}{2\sqrt{2}} \\
&= \frac{1}{2} \\
x &= \frac{1}{4}
\end{aligned}
$$

r

$$
\begin{aligned}
&(x^3+5)(x^3-5) = 704 \\
&x^6-25-704 = 0 \\
&x^6-729 = 0 \\
&(x^3-27)(x^3+27) = 0 \\
&x^3-27 = 0 \quad \text{or} \quad x^3+27 = 0 \\
&x^3 = 27 \quad x^3 = -27 \\
&x = 3 \quad\quad x = -3
\end{aligned}
$$

Question 3

a 12 000

b 12 610 000

c 0.000 26

d 6

e 12 630

Question 4

a After translation, $y = 2^{x+3} = 8 \times 2^x$

b After translation, $y = 3^x - 2$

Question 5

From graph,

$$5^{1.6} \approx 13$$
$$5^{2.4} \approx 13$$
$$5^{2.5} \approx 13$$

Question 6

$$(25 \times 5^x - 1)(5^x - 1) = 0$$

$$\begin{aligned} 25 \times 5^x - 1 &= 0 \\ 5^2.5^x &= 1 \\ 5^x &= \frac{1}{5^2} \\ &= 5^{-2} \\ x &= -2 \end{aligned} \quad \text{or} \quad \begin{aligned} 5^x - 1 &= 0 \\ 5^x &= 1 \\ &= 5^0 \\ x &= 0 \end{aligned}$$

Question 7

If $m = 2^x, m^2 - 5m + 4 = 0$

$(2^x - 1)(2^x - 4) = 0$

$2^x - 1 = 0$ or $2^x - 4 = 0$

$2^x = 1 \qquad 2^x = 4$

$x = 0 \qquad x = 2$

Question 8

$T = ka^t$

$k = 18.9 \qquad (t = 0 \text{ at 10am})$

a

$t = 2, T = 16.3$

$16.3 = 18.9a^2$

$a^2 = \dfrac{16.3}{18.9}$

$a = 0.93$

b

$T = 18.9(0.93)^t$

$32 = 18.9(0.93)^t$

By classpad, $t = -7.26$

7.26 half hours $= 3.63$ hours

$10{:}00 - 3.63 = 6.37$ hours

$= 6{:}22.2$ hours

Time of death $\approx$ 6:22 a.m.